International Journal of Theoretical Physics, Vol. 38, No. 6, 1999

Noncommutative Unification of General Relativity and Quantum Mechanics

Michael Heller[1] and Wiesław Sasin[2]

Received October 26, 1998

We propose a mathematical structure, based on a noncommutative geometry, which combines essential aspects of general relativity with those of quantum mechanics, and leads to correct "limiting cases" of both these physical theories. The noncommutative geometry of the fundamental level is nonlocal with no space and no time in the usual sense, which emerge only in the transition process to the commutative case. It is shown that because of the original nonlocality, quantum gravitational observables should be looked for among correlations of distant phenomena rather than among local effects. We compute the Einstein–Podolsky–Rosen effect; it can be regarded as a remnant or a "shadow" of the noncommutative regime of the fundamental level. A toy model is computed predicting the value of the "cosmological constant" (in the quantum sector) which vanishes when going to the standard spacetime physics.

INTRODUCTION

There are many theoretical indications that at the fundamental level, i.e., below the Planck scale, the manifold structure of spacetime breaks down [there are so many hints scattered in the literature that it is difficult to give a list of references; for a review see Demaret *et al.* (1997)] and that, in particular, time loses its ordinary meaning (e.g., Connes and Rovelli, 1994; Rovelli, 1990, 1991). The problem is that when we give up the manifold structure so many possibilities are open, none of them being more natural than others, that we are left only with our subjective preferences. It seems that the situation has changed with the discovery of noncommutative geometry (Connes, 1994, and references therein). One could claim that the noncommu-

[1] Vatican Observatory, V-00120 Vatican City State; correspondence address: ul. Powstańców Warszawy 13/94, 33-110 Tarnów, Poland; e-mail: mheller@wsd.tarnow.pl.
[2] Institute of Mathematics, Warsaw University of Technology, Plac Politechnikil 00-661 Warsaw, Poland.

tative generalization of the usual geometry is "natural" in the following sense. As is well known, the manifold structure can be defined in terms of the algebra of smooth functions $C^{\infty}(M)$ on a set M (this definition is equivalent to the more standard one in terms of charts and atlases). The algebra $C^{\infty}(M)$ is of course commutative, and to drop the commutativity assumption seems to be a natural step to undertake. Therefore, we take any (associative) algebra and try to see what does happen if we proceed, as closely as possible, along the lines established in the usual differential geometry. It turns out that the change is radical, but, surprisingly, many standard methods can be adapted to the new conceptual context. In this way, many spaces usually regarded as highly pathological (e.g., non-Hausdorff, nonmeasurable) can effectively be investigated with the help of noncommutative methods. Since general algebras are often too difficult to deal with one looks for their suitable representation, and for the algebras in question the standard representation is provided by the algebra of operators in a Hilbert space. And here the chain of our motivations closes up. The idea of noncommutativity appeared first in physics as the noncommutativity of quantum mechanical observables represented by operators in a Hilbert space.

Another attractive thing, from our point of view, about noncommutative spaces is their "global character." In a differentiable manifold M the existence of points is equivalent to the existence of smooth functions which vanish at these points; algebraically such functions form maximal ideals in the algebra $C^{\infty}(M)$ of all smooth functions on M. In noncommutative algebras, in general, there are no maximal ideals, and the concept of point is replaced by that of pure state, which is also familiar from quantum mechanics. In spite of this, a true dynamics can be done on noncommutative spaces, for instance, in terms of derivations of a given noncommutative algebra. It is then evident that when one uses such spaces to model physical processes at the fundamental level the usual idea of spacetime is replaced by something drastically different, but still workable. By restricting the noncommutative algebra to its center we naturally go to the commutative case. In this way, we can recover the standard spacetime geometry.

The above attractive features of noncommutative geometries, and—last but not least—tangible successes in putting into the noncommutative framework the standard model of fundamental interactions (Connes and Lott, 1990), have motivated several attempts at creating a conceptual basis for the noncommutative quantum theory of gravity (e.g., Chamseddine *et al.*, 1993; Sitarz, 1994; Hajac, 1996; Connes, 1996; Chamseddine and Connes, 1996a, b; Madore and Mourad, 1996). All these attempts explicitly or tacitly assume that it is the *geometry of spacetime* which should be made noncommutative [for a review see Madore (1997)]. In Heller et al. (1997) we proposed a scheme for a noncommutative quantization of gravity the main strategy of which consists in starting, from the very beginning, with an abstract noncommutative space and obtaining from it the usual spacetime geometry via the correspondence with the classical case. To describe our idea, let us mention that there is a standard method of obtaining a noncommutative space from a given commutative one (Connes, 1994, pp. 99–102). Roughly speaking, a given commutative space, for instance, a manifold, should be represented as the quotient of a groupoid G by an equivalence relation (which can be given as the action of a group on G), and then one should apply the standard method of constructing a C^*-algebra $\mathcal{A}$ on G. This C^*-algebra is, in general, noncommutative and can serve as a basis for our noncommutative geometry. In the case of spacetime M, one should notice that M can be given as the quotient $M = E/SO(3,1)$, where E is the total space of the fiber bundle of frames over M. The point is that, by taking the Cartesian product $E \times SO(3, 1)$, we obtain a groupoid, the so-called *groupoid of transformations*, and the above method can be directly applied. This is described in Section 1. There is one important proviso: We start, right from the beginning, with the groupoid $G = E \times \Gamma$, where E is a suitable space and Γ a suitable group, forgetting about spacetime M, our aim being to obtain spacetime when going from our noncommutative geometry to the commutative case.

If we assume that E is a smooth manifold and Γ a Lie group, then the noncommutative space corresponding to the C^*-algebra $\mathcal{A}$ is strongly Morita equivalent to the smooth manifold $M = E/\Gamma$. Since in noncommutative geometry, strong Morita equivalence plays the role of isomorphism [for definition see Madore (1995), p. 140, or Masson (1996), p. 40], it would seem that we have gained nothing with our construction. But this is not so. Strong Morita equivalence, being a "noncommutative isomorphism," does not know about points and their neighborhoods, and consequently the noncommutative space based on the algebra $\mathcal{A}$ is equivalent to a smooth manifold "modulo local properties," and can be used in physics to model nonlocal processes at the fundamental level.

Our generalization can go even further if we give up the assumption that E is a smooth manifold. For instance, we could think of E as the total space of a generalized fiber bundle over a spacetime with singularities such that the fibers over "singular points" need not be diffeomorphic to the typical fiber. Even such fibers are admitted which are reduced to the single point. Heller and Sasin (1996) showed that also in such cases the groupoid G is quite a regular space, and our construction can proceed essentially with no changes. However, as a result we obtain a noncommutative space which is no longer strongly Morita equivalent to a manifold, but rather to a space which can be interpreted as spacetime with singularities (Heller *et al.*, 1997). Although this case seems to be more mathematically interesting and more promising from the point of view of physics, in the present paper we shall be concerned with the case where all commutative spaces involved are assumed to be smooth manifolds. Our motivation for doing so is that in the present paper we want to introduce our approach to unifying quantum mechanics with gravity as simply as possible, and going beyond the manifold category would provoke many questions which at the introductory stage would make things more complicated rather than smoothing them out.

Our approach resembles unification theories of the Kaluza–Klein type [for a noncommutative version of such a theory see Madore (1995), p. 180] with the equivalence classes of the groupoid fibers (see below Section 5) playing the role of "internal spaces" in Kaluza–Klein theories. It can be regarded as describing some finite-dimensional system with internal degrees of freedom responsible for quantum gravity effects. Let us notice, however, that the system in question could be a "gravitating particle" (as when we consider the Einstein–Podolsky–Rosen experiment in Section 8), or a cosmological model in a quantum gravity regime (if we "lift" a cosmological solution from spacetime M to the groupoid G; see end of Section 3). This largely depends on the choice of the group Γ. For this reason we prefer to speak of our approach as a step toward a unification of general relativity with quantum mechanics, although by a slight abuse of language we shall also occasionally speak about quantization of gravity.

The organization of our material is the following. In Section 1, we construct the Hilbert space for our model. Section 2 summarizes those aspects of noncommutative geometry which are necessary to formulate (in Section 3) a noncommutative version of general relativity. Its quantization scheme is presented in Section 4, and the transitions to the usual spacetime geometry, on one hand and to the standard quantum mechanics on the other hand are discussed in Section 5. In Section 6, we check the consistency of our scheme by computing a simple model in which the groupoid G is a Cartesian product of the total space of the frame bundle over a 3-dimensional Minkowski spacetime by the finite group D_4. In Section 7, we demonstrate that, in our scheme observable quantum gravitational phenomena should be looked for among correlations between distant measurements rather than among "local phenomena," and in Section 8 we show that the Einstein–Podolsky–Rosen experiment finds its natural explanation in our approach. It can be regarded as a remnant or shadow of the "totally nonlocal" noncommutative regime. Section 9 summarizes the main advantages of our model and stresses some of its weaker points. Some overlaps with the material presented in (Heller et al., 1997) are indispensable not only to make the present paper self-consistent, but also because our aim is to discuss more carefully physical aspects of our approach than was possible in Heller *et al.* (1997), in which the mathematical foundations were the primary objective.

1. GROUPOID OF FUNDAMENTAL SYMMETRIES

In this section we construct the Hilbert space for our model. We start from the direct product $G = E \times \Gamma$ where E is an n-dimensional smooth manifold and Γ a Lie group acting on E (to the right). Elements of Γ are "fundamental symmetries" of our scheme. Heuristically, we could think of E as the total space of the fiber bundle of frames over spacetime M, and of Γ as its structural group (a connected component of the Lorentz group). However, we insist upon starting just from a manifold E and a group Γ of "fundamental symmetries," our aim being to deduce spacetime M from our model via the correspondence with macroscopic physics. In the present paper we leave the group Γ unspecified. The correct choice of Γ is left for the future development of the proposed model, and it should be made on physical grounds. Moreover, it can turn out that our scheme is too narrow to incorporate all required physics; in such a case the scheme could be enlarged by substituting for Γ a supergroup or a quantum group (and suitably modifying the model; preliminary analysis shows that it is possible).

Our next step will be to regard G as a groupoid. Roughly speaking, groupoid differs from group by the fact that not all its elements can be composed with each other (composition can be done only within certain subsets of the groupoid). For the precise definition of groupoid see, for instance, Renault (1980); here we give a less formal description of G as a groupoid.

Of course, G is a set of pairs $\gamma = (p, q)$ where $q = pg$, $p, q \in E$, $g \in \Gamma$ [one can also write $\gamma = (p, g)$, $g \in \Gamma$]. We can think of such a pair as an arrow starting at p and ending at q. This arrow can be interpreted as a fundamental symmetry operation (the name "fundamental symmetry" can be attributed, by only a slight abuse of language, to both elements of Γ and elements of G). For G to be a groupoid a subset $G^{(0)}$ of it should be distinguished, namely the subset of all elements of G of the from (p, e), $p \in E$, where e is the neutral element of Γ, and there must exist a composition law such that elements γ_1 and γ_2 of G, viewed as arrows, can be composed with each other, $\gamma = \gamma_1 \circ \gamma_2$, if the end of γ_2 coincides with the beginning of γ_1. To formally express properties of the composition one introduces two following mappings: the *source mapping* s: $G \to G^{(0)}$ defined by $s(p, q) = p$, and the *range mapping* r: $G \to G^{(0)}$ defined by $r(p, q) = q$. Then one defines $G^{(2)} := \{(\gamma_1, \gamma_2) \in G \times G: s(\gamma_1) = r(\gamma_2)\}$, and some natural conditions are postulated, for instance, $s(\gamma_1 \circ \gamma_2) = s(\gamma_2)$ and $r(\gamma_1 \circ \gamma_2) = r(\gamma_1)$ for every $\gamma_1, \gamma_2 \in$

$G^{(2)}$. These conditions can be easily read from the diagram presenting the composition $\gamma = \gamma_1 \circ \gamma_2$ in the form of arrows (Connes, 1994, pp. 99–100). We should also notice that each $\gamma \in G$ has the two-sided inverse γ^{-1} such that $\gamma\gamma^{-1} = r(\gamma)$ and $\gamma^{-1}\gamma = s(\gamma)$ (for brevity we omit the composition symbol $\circ$). G has the natural structure of a fibered space with the fibers $G_p = \{p\} \times \Gamma$, $p \in E$.

The groupoid G with the above structure is also called the *semi-direct product* of E and Γ, and denoted by $G = E \rtimes \Gamma$. The groupoid G is said to be *smooth* if G and $G^{(0)}$ carry differentiable structures such that the mappings s and r are submersions, and the composition mapping $\circ$: $G^{(2)} \to G$ and the natural inclusion mapping ι: $G^{(0)} \to G$ are smooth. In our case, G is evidently a smooth groupoid.

Now, our strategy is the following. First, we shall try to construct, based on G, a noncommutative differential geometry. To this end we define the algebra $\mathcal{A} = C_c^\infty(G, \mathbf{C})$ of smooth, compactly supported, complex-valued functions on G with the convolution

$$(a * b)(\gamma) := \int_{G_p} a(\gamma_1)b(\gamma_2)$$

as multiplication, where $a, b \in \mathcal{A}$, and $\gamma = \gamma_1\gamma_2$; $\gamma, \gamma_1, \gamma_2 \in G_p$; $p \in E$. If Γ is a non-Abelian group, the convolution is noncommutative, giving rise to a noncommutative geometry (which we shall construct in the next section). The above integral is taken with respect to the (left) invariant Haar measure. $\mathcal{A}$ is also an involutive algebra with involution defined as $a^*(\gamma) = a(\gamma^{-1})$. Second, based on the geometry determined by the algebra $\mathcal{A}$, we shall first define a generalized Einstein's equation (in the operator form), and then, on each fiber G_q, define square-integrable functions equipped with the suitable Hilbert space structure. The direct sum $\mathcal{H} = \bigoplus_{q \in E} L^2(G_q)$ will serve as a state space of our quantum mechanics. The modulus squared $|\psi|^2$ of the "wave function" $\psi \in L^2(G_q)$ is the probability density of the "fundamental symmetry" $\gamma \in G_q$ to occur.

The crucial point is to make the noncommutative geometry based on the algebra $\mathcal{A} = C_c^\infty(G, \mathbf{C})$ and the Hilbert space $\mathcal{H} = \bigoplus_{q \in E} L^2(G_q)$ to collaborate with each other. This will be achieved in the following way. After solving the generalized Einstein equation, we complete the algebra $\mathcal{A}$ to a C^*-algebra, and then we find a representation of this algebra in the Hilbert space $\mathcal{H}$. Now, we can develop the quantization scheme by following either the standard formalism of bounded operators on a Hilbert space or the C^*-algebra approach. We shall describe all stages of the above scheme in the following sections.

2. NONCOMMUTATIVE GEOMETRY OF THE GROUPOID

2.1. Differential Algebra

As was demonstrated by Koszul (1960) and later extensively used by others, the differential geometry on a manifold M can be done in terms of the algebra $C^{\infty}(M)$ of smooth functions on M and the $C^{\infty}(M)$-modules of smooth sections of smooth vector bundles over M. The main idea of generalizing the standard differential geometry is to replace the commutative algebra $C^{\infty}(M)$ by any, not necessarily commutative, associative algebra. In this way, one obtains a vast generalization of the traditional geometry but, unfortunately, the generalization is not unique: at several crucial points one can proceed in various directions, thus obtaining different versions of noncommutative differential geometry [for a penetrating discussion, see Dubois-Violette (1995)]. Happily enough, if we choose the derivation-based version of differential geometry on the smooth groupoid $G = E \times \Gamma$, the generalization is practically unique. The structure of G turns out to be simple enough to exclude unnecessary complications and at the same time rich enough to guarantee interesting results. The derivation-based calculus has been developed in many works (e.g., Dubois-Violette, 1988, 1995; Dubois-Violette *et al.*, 1989a, b, 1990; Dubois-Violette and Michor, 1994; Sasin and Heller 1995). In the rest of this section we follow the last of these references.

Derivation of the algebra $\mathcal{A}$ is defined to be a linear transformation (endomorphism) $v: \mathcal{A} \to \mathcal{A}$ satisfying the Leibniz rule

$$v(ab) = v(a)b + bv(a)$$

$a, b \in \mathcal{A}$. The set of all derivations of $\mathcal{A}$ is denoted by $\mathrm{Der}\mathcal{A}$. It is a Lie algebra with respect to the bracket operation $[u, v] = uv - vu$, $u, v \in \mathrm{Der}\mathcal{A}$. In the case of the algebra $C^{\infty}(M)$, $\mathrm{Der}(C^{\infty}(M))$ is a $C^{\infty}(M)$-module, and it corresponds to all vector fields on M. In the case of a noncommutative algebra $\mathcal{A}$, $\mathrm{Der}\mathcal{A}$ is not, in general, an $\mathcal{A}$-module, but only a $\mathfrak{Z}(\mathcal{A})$-module, where $\mathfrak{Z}(\mathcal{A})$ denotes the center of $\mathcal{A}$ (i.e., the set of all elements of $\mathcal{A}$ which commute with all elements of $\mathcal{A}$). Although $\mathrm{Der}\mathcal{A}$ can be thought of as a noncommutative counterpart of vectors fields, it should be remembered that in the framework of noncommutative geometry, "vector fields" are in general global objects and consequently they cannot be said to consist of vectors.

The pair $(\mathcal{A}, V)$, where V is a $\mathfrak{Z}(\mathcal{A})$-submodule of $\mathrm{Der}\mathcal{A}$, is called *differential algebra*. In our case $\mathcal{A} = C_c^{\infty}(G, \mathbf{C})$, and as V we choose those derivations of $\mathcal{A}$ which are naturally adapted to the structure of $G = E \times \Gamma$ (as a direct product), i.e., all those $v \in \mathrm{Der}\mathcal{A}$ which can be represented in the form $v = v_E + v_\Gamma$, where v_E is the "component" of v parallel to E, and v_Γ the "component" of v parallel to Γ. More formally, $v \in V$ is said to be *parallel to E* if, for any $\alpha \in C^{\infty}(\Gamma)$, $v(\alpha \circ pr_\Gamma) = 0$, where pr_Γ is the obvious projection. The set of all derivations of $\mathcal{A}$ parallel to E is denoted by V_E. Derivations *parallel* to Γ, denoted by V_Γ, are defined analogously. Therefore,

$$V = V_E \oplus V_\Gamma$$

2.2. Metric Structure

To proceed further, we must introduce a *metric*, i.e., a $\mathcal{Z}(\mathcal{A})$-bilinear nondegenerate symmetric mapping $g: V \times V \to \mathcal{A}$. We chose the metric in the form

$$g = pr_E^* g_E + pr_\Gamma^* g_\Gamma \tag{1}$$

where g_E and g_Γ are metrics on E and Γ, respectively. The above choice of both V and g is the simplest and the most natural one (it is naturally adapted to the product structure of $G = E \times \Gamma$).

In general, to define a metric in a noncommutative case is a delicate matter. It turns out that, in sharp contrast to the standard geometry, in a broad case of noncommutative differential calculi there exists an essentially unique metric structure. It has been shown by Madore and Mourad (1996) (see also Madore, 1997) that for any derivation-based noncommutative differential calculus, such that all derivations are internal, there exists a unique metric (modulo a nonsingular matrix). In such a case, the metric coefficients must lie in the center of a given algebra, and in the commutative limit they cannot be functions of coordinates. "In the commutative case (. . .), each differential calculus determines a Stehbein [a global field of moving frames] and thereby a metric. In the commutative limit all of the noncommutative differential calculi are either singular (. . .), or have a common limit. The moving frame however and the associated metric remain as a shadow of the noncommutative structure" (Madore and Mourad, 1996, p. 433). This is of course the consequence of the nonlocal character of noncommutative geometry.

Let us notice that the Γ part $pr_\Gamma^* g_\Gamma$ of metric (1) is the pullback of the (Riemann) metric g_Γ on the group Γ and, in agreement with Madore and Mourad (1996), it is essentially unique. However, the E part $pr_E^* g_E$ of metric (1) is in fact determined by the Lorentz metric on the spacetime M (since $g_E = \pi_M^* g_M$, where $\pi_M: E \to M$ is the canonical projection), and consequently there are as many metrics $pr_E^* g_E$ as there are Lorentz metrics on M. These facts are important as far as our construction of "noncommutative general relativity" is concerned (see Section 3).

Notice that the differential geometry can be done in terms of the algebra $\mathcal{A}$ even if the metric pr_E^* is *weakly degenerate*, i.e., if it satisfies the following condition:

$$\forall v \in V_E: \quad (\forall u \in V_E: \quad \tilde{g}(u, v) = 0)$$
$$\Rightarrow (\forall a \in (\pi_M \circ pr_E)^*(C^\infty(M)): \quad v(a) = 0)$$

It can be shown that if $g_E = \pi_M g$, where g is a (Lorentz) metric on M, then $pr_E^* g_E$ is weakly degenerate. Indeed, the condition

$$\forall Y \in \chi(M): \quad ((\forall X \in \chi(M): \quad g(X, Y) = 0) \Rightarrow (Y = 0))$$

is equivalent to

$$\forall \bar{Y} \in V_E: \quad ((\forall \bar{X} \in V_E: \quad pr_E^* g_E(\bar{X}, \bar{Y}) = 0)$$
$$\Rightarrow \forall a \in (\pi_M \circ pr_E)^*((C^\infty(M)): \quad \bar{Y}(a) = 0)$$

where $\bar{X} \in V_E$ is the canonical lift of $X \in \mathcal{X}(M)$, and similarly for $\bar{Y} \in V_E$. The remark that each $v \in V_E$ is of this form ends the proof.

2.3. Connection and Curvature

Now let us define the mapping Φ_g: $V \to V^*$, where V^*, the dual of V, is the set of $\mathcal{Z}(\mathcal{A})$-homomorphisms from V to $\mathcal{A}$, by

$$\Phi_g(u)(v) = g(u, v)$$

$u, v \in V$. The mappings Φ_g and Φ_g^{-1} play roles analogous to those of lowering and raising indices in the standard tensorial calculus. The set V^+ such that $\Phi_g^{-1}(V^+) = V$ is the set of "invertible forms." In our case, all forms are invertible, i.e., $V^+ = V^*$.

Now we define the *preconnection* ∇^*: $V \times V \to V^*$ with the help of the usual Koszul formula

$$(\nabla_u^* v)(x) = \frac{1}{2}[u(g(v, x)) + v(g(u, x)) - x(g(u, v))$$
$$+ g(x, [u, v]) + g(v, [x, u]) - g(u, [v, x])]$$

for $u, v, x \in V$, and the *linear connection* ∇: $V \times V \to V$ by

$$\nabla_u v = \Phi_g^{-1}(\nabla_u^* v)$$

The *curvature* of this connection is the operator R: $V^3 \to V$ defined by

$$R(u, x)y = \nabla_u \nabla_x y - \nabla_x \nabla_u y - \nabla_{[u,x]} y$$

If V is a *free* $\mathcal{Z}(\mathcal{A})$-module, we can choose a basis in it and for any linear operator T: $V \to V$ define the *trace* of T in the usual way, $\mathrm{tr}T = \Sigma_{i=1}^k T_i^i$. Let us notice that V_E is always a free module. Whether V_Γ is a free module depends on the group Γ. If $V = V_E \oplus V_\Gamma$ is not a free module, we can proceed as in Sasin and Heller (1995) (for the case where Γ is a finite group see Section 6). In the following, we shall, for simplicity, assume that V is a free $\mathfrak{Z}(\mathcal{A})$-module. In such a case, for any fixed pair x, $y \in V$, we define the family of operators R_{xy}: $V \to V$ by

$$R_{xy}(u) = R(u, x)y$$

The *Ricci curvature* is $\mathbf{ric}(x, y) = \mathrm{tr}R_{xy}$. Finally, by putting $\mathbf{ric}(x, y) = g(\mathbf{R}(x), y)$ we obtain the *Ricci operator* $\mathbf{R}$: $V \to V$ [$\mathbf{R}$ is the adjoint operator of the $\mathfrak{Z}(\mathcal{A})$-bilinear form $\mathbf{ric}$: $V \times V \to \mathfrak{Z}(\mathcal{A})$]. Now we have at our disposal all the necessary tools to define a noncommutative version of Einstein's equation.

3. NONCOMMUTATIVE GENERAL RELATIVITY

Let us define the *generalized Einstein equation* in the operator form

$$\mathbf{R} - \frac{1}{2\alpha} r\mathbf{I} + \Lambda\mathbf{I} = \kappa\mathbf{T} \tag{2}$$

where $\alpha = \mathrm{tr}\mathbf{I}$, $r = \mathrm{tr}\mathbf{R}$, Λ and κ are constants related to the cosmological constant and Einstein's gravitational constant, respectively, and $\mathbf{T}$ is a suitably generalized energy-momentum operator. Since it could be expected that at the fundamental level there is only "pure noncommutative geometry" we assume that $\mathbf{T} = 0$, but for the sake of generality we keep Λ in the equation (if necessary we can always put $\Lambda = 0$). Therefore, the generalized Einstein equation assumes the form

$$\mathbf{G} = 0 \tag{3}$$

where $\mathbf{G} := \mathbf{R} + 2\Lambda\mathbf{I}$. It can be easily seen that the set $\ker\mathbf{G} := \{v \in V: \mathbf{G}(v) = 0\}$ is a $\mathfrak{Z}(\mathcal{A})$-submodule of V. The differential algebra $(\mathcal{A}, \ker\mathbf{G})$, where $\mathcal{A} = C_c^\infty(G, \mathbf{C})$, can be regarded as a solution of the generalized Einstein equation (strictly speaking only $\ker\mathbf{G}$ is determined by this equation).

Because of the form of metric (1) the generalized Einstein equation can be written as

$$\mathbf{G}_E + \mathbf{G}_\Gamma = 0$$

where $\mathbf{G}_E$ is the part parallel to E, and $\mathbf{G}_\Gamma$ is the part parallel to Γ. Since in the Γ direction there is essentially one metric (see Section 2.2), the equation $\mathbf{G}_\Gamma = 0$ should be solved for derivations, i.e., we should look for $v \in \ker\mathbf{G}_\Gamma \subset V_\Gamma$

On the other hand, since the equation $\mathbf{G}_E = 0$ is essentially a "lifting" of the standard Einstein equation in the spacetime M, all derivations $v \in V_E$ satisfy it (all derivations $v \in V_\Gamma$ satisfy it trivially), i.e., $\mathbf{G}(v) = 0$ for every $v \in V_E$. Therefore, we can write

$$pr_E^* g_E(\mathbf{G}(V), u) = \mathbf{ric}(u, v) = 0$$

for every $u, v \in V_E$. This equation is equivalent to the Einstein equation in spacetime M, and it should be solved for $pr_E^* g_E$.

To solve equation (3) [or equation (2)] is a difficult task, but we can use the above "lifting property" to show that it has many solutions. Indeed, every "lifted" solution of the usual Einstein equation solves the generalized Einstein equation. Let $C^\infty(M)$ and $C_c^\infty(G, \mathbf{C})$ be the algebras corresponding to the original solution and to the "lifted" solution, respectively. These algebras are strongly Morita equivalent. This means that from the point of view of noncommutative geometry they contain the same information [let us notice, however, that $C_c^\infty(G, \mathbf{C})$ ignores local properties of $C^\infty(M)$]. Of course, not all solutions of the generalized Einstein equation are generated in this way.

4. QUANTIZATION OF NONCOMMUTATIVE GENERAL RELATIVITY

In the present section we shall consider the differential algebra $(\mathcal{A}, \ker\mathbf{G})$. Recall that $\mathcal{A}$ is an involutive algebra with the involution defined as $a^*(\gamma) = (\gamma^{-1})$, $a \in \mathcal{A}$, $\gamma \in G$, and the convolution $(a * b)(\gamma) = \int_{G_p} a(\gamma_1)b(\gamma_2)$ as multiplication. Now, our aim is to extend $\mathcal{A}$ to a C^*-algebra and quantize it with the help of the standard algebraic method (e.g., Thirring, 1979).

By applying the theorem proved by Connes (1979) to our case, we learn that the involutive algebra $\mathcal{A} = C_c^\infty(G, \mathbf{C})$, for each $q \in G^{(0)}$, has the representation π_q in a Hilbert space $\mathcal{H} = L^2(G_q)$

$$\pi_q\colon \quad \mathcal{A} \to \mathcal{B}(\mathcal{H})$$

where $\mathcal{B}(\mathcal{H})$ denotes the algebra of bounded operators on $\mathcal{H}$, given by

$$(\pi_q(a)\psi)(\gamma) = \int_{G_q} a(\gamma_1)\psi(\gamma_1^{-1}\gamma) \tag{4}$$

$\gamma = \gamma_1 \circ \gamma_2$, $\gamma, \gamma_1, \gamma_2 \in G_q$, $\psi \in L^2(G_q)$, $a \in \mathcal{A}$, and that the completion of $\mathcal{A}$ with respect to the norm

$$\|a\| = \sup_{q \in G^0} \|\pi_q(a)\|$$

is a C^*-algebra. We shall denote it by $\mathcal{E}$ and call it an *Einstein C^*-algebra*.

Now we can formulate postulates of our noncommutative theory of quantum gravity.

Postulate 1. A quantum gravitational system is represented by an Einstein C^*-algebra $\mathcal{E}$, and its observables by Hermitian elements of $\mathcal{E}$ (the set of all Hermitian elements of $\mathcal{E}$ will be denoted by $\mathcal{E}_H$).

We speak of "observables" in the quantum gravity regime by analogy with the standard quantum mechanics. Whether these "observables" leave traces in the macroscopic world remains to be seen (we come back to this question in Section 7). By the same analogy, we can say that the spectrum of a Hermitian element of $\mathscr{E}$ represents possible measurement results of this observable.

Postulate 2. Let $\mathscr{S}$ denote the set of all states of the algebra $\mathscr{E}$; elements of $\mathscr{S}$ represent states of the system, and pure states of $\mathscr{E}$ represent pure states of the system.

Postulate 3. If $a \in \mathscr{E}_H$ and $\phi \in \mathscr{S}$, then $\phi(a)$ is the expectation value of the observable a when the system is in the state ϕ.

Recall that *states* of $\mathscr{E}$ are defined to be positive linear functionals ϕ on $\mathscr{E}$ such that $\|\phi\| = 1$. Convex combinations of states are states. A state which cannot be expressed as a convex combination of other states is said to be a *pure state.*

Whereas the above three postulates are in close analogy with the standard C^*-algebraic approach to quantum mechanics, the following fourth postulate is a new ingredient of our scheme.

Postulate 4. The dynamical equation of the system, described by $\mathscr{E}$, is

$$i\hbar\pi_q(v(a)) = [\pi_q(a), F] \tag{5}$$

for every $q \in G^{(0)}$. We postulate that here $v \in \ker\mathbf{G}$, and in this way the generalized Einstein equation (3) is coupled to the quantum dynamical equation (5). F is a an operator, F: $\mathscr{H} \to \mathscr{H}$, which in the transition to the commutative case should reproduce the Hamilton operator (see Section 5). Together with the group Γ, the operator F is a "free entry" of our quantization scheme. It should be specified on physical grounds. To solve equation (5) means to find $a \in \mathscr{E}$ such that, for every $v \in \ker\mathbf{G}$, $\pi_q(v(a))$ gives the same result as $-(i/\hbar)[\pi_q(a), F]$ when acting on $\psi \in L^2(G_q)$.

Equation (5) is a noncommutative counterpart of the Schrödinger equation in the Heisenberg picture of the usual quantum mechanics,

$$i\hbar\left(\frac{d}{dt}\hat{A}(t)\right) = [\hat{A}(t), H]$$

where H is the Hamilton operator. In this picture, state vectors are independent of time and all time dependence goes into the operators. Since in the noncommutative framework the standard concept of time breaks down, the dynamics of the system is expressed in terms of derivations of the Einstein algebra.

Equation (5) acts on the Hilbert space $L^2(G_q)$. This space should be regarded as a counterpart of the Hilbert space in the position representation of quantum mechanics. However, now the "position space" is more abstract: the quantity $|\psi(\gamma)|^2$ is the probability density of the "fundamental symmetry" $\gamma \in G_q$ to occur.

It might turn out that in order to make equation (5) manageable we would have to impose on it some further conditions, for instance, to assume that the triple $(\mathcal{A}, \mathcal{H}, F)$ is a Fredholm module or to assume some of its summability properties (Connes, 1994, pp. 288–291; Masson, 1996, pp. 117–120).

To substantiate our approach we should show that it reproduces, in the commutative limit, the usual general relativity and the usual quantum mechanics. We shall demonstrate this in the next section.

5. TRANSITION TO GENERAL RELATIVITY AND QUANTUM MECHANICS

The "canonical way" of obtaining a commutative geometry from a noncommutative one is to restrict the corresponding noncommutative algebra $\mathcal{A}$ to its center $\mathfrak{Z}(\mathcal{A})$. This can be done in the following way.

Let $\mathfrak{Z}(\mathcal{A})^{\#}$ be the set of all characters of $\mathfrak{Z}(\mathcal{A})$, i.e., the set of all *-homomorphisms from $\mathfrak{Z}(\mathcal{A})$ to $\mathbf{C}$. On the strength of the Gel'fand theorem, the algebra $\mathfrak{Z}(\mathcal{A})^{\#}$ is isomorphic with the algebra of continuous functions on the groupoid G (with the usual multiplication). This algebra is given by the Gel'fand representation

$$\rho^{\mathfrak{Z}(\mathcal{A})}: \quad \mathfrak{Z}(\mathcal{A}) \to \mathbf{C}^{\mathfrak{Z}(\mathcal{A})^{\#}}$$

defined by

$$\rho^{\mathfrak{Z}(\mathcal{A})}(a)(\chi) = \chi(a)$$

where $a \in \mathfrak{Z}(\mathcal{A})$, $\chi \in \mathfrak{Z}(\mathcal{A})^{\#}$, and $\mathfrak{Z}(\mathcal{A})^{\#}$ can be identified with G [in general $G \subset \mathfrak{Z}(\mathcal{A})^{\#}$]. The algebra $\rho^{\mathfrak{Z}(\mathcal{A})}(\mathfrak{Z}(\mathcal{A}))$ consists of continuous functions on G, but since G is a smooth manifold, we can assume that these functions are smooth (if necessary we can restrict this algebra to the subalgebra of smooth functions) (Palais, 1981). We shall denote the algebra of these functions by G^{∞}. As is well known, there is a bijection

$$\mathfrak{Z}(\mathcal{A})^{\#} \to \mathrm{Spec}\mathfrak{Z}(\mathcal{A})$$

where $\mathrm{Spec}\mathfrak{Z}(\mathcal{A})$ denotes the set of maximal ideals of $\mathfrak{Z}(\mathcal{A})$, given by

$$\chi \mapsto \ker\chi$$

$\chi \in \mathfrak{Z}(\mathcal{A})^{\#}$. Each maximal ideal $\ker\chi$ determines a point of G (such a point is given by the set of functions belonging to G^{∞} vanishing at this point). We recall that points of G are "fundamental symmetries" of our theory. In this way, the full geometry of the groupoid is given by the pair (G, G^{∞}).

Since the prototype of our groupoid was the Cartesian product $G = E \times \Gamma$ (see Introduction), where E was supposed to be the total space of the frame bundle over spacetime M, we recover M by forming, first, the quotient $E = G/\Gamma$, and then $M = E/\Gamma$ [or $M = (E/\Gamma)/\Gamma$; see below for the detailed construction]. The generalized Einstein equation (2) "projected down," in this way, to spacetime M gives the usual Einstein equation of general relativity.

Let us assume that Γ is a compact group. In this case, there is a simpler way of going to the classical case. Two fibers G_p and G_q, $p, q \in E$, are said to be *equivalent* if there is $g \in \Gamma$ such that $q = pg$. The set of all functions belonging to the algebra $\mathcal{A}$ which are constant on the equivalence classes of fibers of the above equivalence relation form a subalgebra $\mathcal{A}_{\text{proj}}$ of *projectible functions*. It can be easily seen that $\mathcal{A}_{\text{proj}} \subset \mathcal{Z}(\mathcal{A})$. Consequently, if $f, g \in \mathcal{A}_{\text{proj}}$, then the convolution $f * g$ reduces to the usual multiplication, and $\mathcal{A}_{\text{proj}}$ is isomorphic with the algebra of smooth functions on the manifold M. In this way, by restricting $\mathcal{A}$ to $\mathcal{A}_{\text{proj}}$ we obtain the standard spacetime geometry.

Now we discuss the transition from our noncommutative theory to the usual quantum mechanics. To do this we must look more carefully at equation (5). Let us assume that $\ker\mathbf{G} \neq 0$, and let us denote by $\mathcal{E}_{\mathbf{G}}$ the set of all solutions of equation (5), i.e.,

$$\mathcal{E}_{\mathbf{G}} = \{a \in \mathcal{E}\colon i\hbar\pi_q(v(a)) = [\pi_q(a), F], \forall v \in \ker\mathbf{G}\}$$

$\mathcal{E}_{\mathbf{G}}$ is clearly a $\mathbf{C}$-linear space. We shall demonstrate that if $a \in \mathcal{E}_{\mathbf{G}}$, then $[\pi_q(a), F] = 0$. Indeed, let us define the mapping

$$\hat{da}\colon \quad \ker\mathbf{G} \to \mathcal{B}(\mathcal{H})$$

by

$$(\hat{da})(v) = \pi_q(v(a))$$

It can be easily demonstrated that $\hat{da}$ is $\mathbf{C}$-linear. Therefore, for every $v \in \ker\mathbf{G}$ one has

$$(\hat{da})(v) = \frac{1}{i\hbar}[\pi_q(a), F]$$

The right-hand side of this equation does not depend on v; therefore $\hat{da}$ is constant, and because of linearity $\hat{da}(0) = 0$ implies $\hat{da} = 0$. Hence, for every $a \in \mathcal{E}_G$

$$[\pi_q(a), F] = 0 \tag{6}$$

as claimed.

We see that in the quantum gravity regime, when quantum effects are coupled to gravity (i.e., when $v \in \ker\mathbf{G}$), our dynamical equation (5) assumes the form (6), which basically says that $\pi_q(a)$ is a "constant of motion." This seems reasonable since in the noncommutative era there is no time in the usual sense. However, it is essential to assume, from the very beginning, the validity of equation (5) rather than (6); this is necessary to obtain the correct commutative case. Indeed, let us assume that the gravitational field is weak so that quantum gravity effects can be neglected. This means that we omit the assumption that $v \in \ker\mathbf{G}$ (i.e., general relativity decouples from quantum mechanics). In such a case, for a given $v \in \mathrm{Der}\mathcal{A}$ the above proof of equality (6) is no longer valid, and we come back to the original equation (5). For simplicity, let us assume that Γ is compact. First, we shall show that a derivation $v \in V$, $v \neq 0$, satisfies equation (5) for all $a \in \mathcal{A}_{\text{proj}}$ if and only if $v \in V_\Gamma$. Indeed, since $\mathcal{A}_{\text{proj}} \subset \mathcal{Z}(\mathcal{A})$, $[\pi_q(a), F] = 0$ and this implies $i\hbar\pi_q(v(a)) = 0$. Hence, we have $v(a) = 0$, and consequently $v \in V_\Gamma$.

Therefore, in this case, equation (5) acts only vertically. This means the following. Let $(p_1, g_1), (p_2, g_2) \in G$, $p_1, p_2 \in E$, $g_1, g_2 \in \Gamma$. We define the equivalence relation $(p_1, g_1) \sim (p_2, g_2) \Leftrightarrow \exists_{g\in\Gamma}\, p_2 = p_1 g$. Then we consider only those "wave functions" $\tilde{\psi}$ which have the following invariance property: $\tilde{\psi}(p_1, g_1) = \tilde{\psi}(p_2, g_2)$ ($\tilde{\psi}$ is constant on equivalence classes of $\sim$). "Wave functions" with this property will be called Γ-*invariant*, and the set of all Γ-invariant functions will be denoted by $L^2_\Gamma(G)$.

Now, let $v \in V_\Gamma$, and let $\partial/\partial x^i$ be a local basis on $G = E \times \Gamma$, obtained by lifting the local basis on Γ, and chosen so as to obtain $v = \partial/\partial x^i$. If we assume that $F|L^2_\Gamma(G) = H$, where H is the Hamiltonian operator, equation (5) assumes the familiar form

$$i\hbar\, \widehat{\frac{\partial a}{\partial x^i}} = [\hat{a}, H]$$

of the Schrödinger equation in the Heisenberg picture of quantum mechanics (the caret denotes here the image under the π_q representation of the algebra $\mathcal{A}$). This, of course, had to be expected. In the noncommutative quantum regime there are no concepts of points and time instants [space and time are somehow hidden in the subalgebra of $\mathcal{Z}(\mathcal{A})$], and the usual concept of space-time appears only in the transition process to standard physics. No wonder that we obtain the Heisenberg picture in which state vectors are time independent and all time dependance goes into the operators.

6. A SIMPLE EXAMPLE

In this section we analyze a simple model of our scheme. The basis of this model is the groupoid $G = E \times D_4$, where E is the total space of the frame bundle over the three-dimensional Minkowski spacetime M^3 and D_4 is a group consisting of four rotations by the angle $\pi/2$ and four reflections with respect to two directions crossing each other at the origin. If r denotes rotation and s reflection, the following relations are assumed to be satisfied:

$$r^4 = 1, \qquad s^2 = 1, \qquad srs = r^{-1}$$

D_4 is a finite noncommutative subgroup of the group SU(2). D_4 acts on E (to the right) on the plane (x, y) leaving the t axis fixed.

For every vector field $X \in \mathcal{X}(M^3)$ on the Minkowski spacetime M^3, there exists its lifting to $\bar{X} \in \mathcal{X}(G)$ such that the vector field $\bar{X}$ is constant on the fibers (of G) parallel to M^3. All such fields form a $\mathfrak{Z}(\mathcal{A})$-submodule V_E of the $\mathfrak{Z}(\mathcal{A})$-module Der($\mathcal{A}$), where $\mathcal{A} = C^\infty(G)$. Analogously, we have a $\mathfrak{Z}(\mathcal{A})$-submodule V_{D_4} of the $\mathfrak{Z}(\mathcal{A})$-module Der($\mathcal{A}$). Our simple model is based on the differential algebra $(\mathcal{A}, V)$, where $V = V_E \oplus V_{D_4}$, $V \subset \text{Der}(\mathcal{A})$. Accordingly, the algebra $C^\infty(G)$ can be "decomposed" into the algebras $C^\infty(E)$ and $\mathbf{C}[D_4]$, where $C^\infty(E) = \iota_E^*(C^\infty(G))$ and $\mathbf{C}[D_4] = \iota_{D_4}^*(C^\infty(G))$, ι_E and ι_{D_4} being natural embeddings of E and D_4 into G, respectively.

In agreement with formula (1), we assume the metric on the $\mathfrak{Z}(\mathcal{A})$-module V of the form

$$g = pr_E^* \circ \pi_1^* \eta + pr_{D_4}^* g_{D_4}$$

where η and g_{D_4} are the Minkowski metric on M^3 and a metric on the group D_4, respectively; pr_E and pr_{D_4} are the obvious projections from the groupoid, and π_1 is the canonical projection from E to M^3.

Since the "parallel geometry" (geometry based on V_E) is rather obvious (see below), we shall focus on the "vertical geometry" (geometry of D_4). As is well known, the group algebra $\mathbf{C}[D_4]$ can be constructed in the following way:

$$\mathbf{C}[D_4] = \left\{ \sum_{i=1}^{8} c_i A_i \colon c_i \in \mathbf{C},\ A_i \in D_4,\ i = 1, \ldots, 8 \right\}$$

(8 is the rank of D_4), with the usual addition and convolution as multiplication. For any finite group Γ there exists an isomorphism

$$T\colon \quad \mathbf{C}[\Gamma] \to \prod_{i=1}^{k} M_{n_i}(\mathbf{C})$$

where $M_{n_i}(\mathbf{C})$ are $n_i \times n_i$ matrices and i runs over all irreducible representations of Γ, such that $T(\varphi * \psi) = T(\varphi) \cdot T(\psi)$, with the asterisk denoting convolution and the dot the usual matrix multiplication. The group D_4 has four irreducible representations of rank 1, and one irreducible representation of rank 2 (Serre, 1978). Therefore, in our case, the above isomorphism assumes the form

$$T: \quad \mathbf{C}[D_4] \to \mathbf{C} \oplus \mathbf{C} \oplus \mathbf{C} \oplus \mathbf{C} \oplus M_2(\mathbf{C})$$

given by

$$T = (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \rho^1)$$

where $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ are the rank 1 irreducible representations of D_4, and ρ^1 is the rank 2 irreducible representation of D_4.

The set of derivations $\mathrm{Der}(\mathbf{C}[D_4])$ of the algebra $\mathbf{C}[D_4]$ is isomorphic with $\mathrm{Der}(M_2(\mathbf{C}))$. It can be shown by straightforward computation that if (e_α), $\alpha = 1, \ldots, n^2 - 1$, is a basis of the Lie algebra su(n) of the Lie group SU(n), and if $c^\gamma_{\alpha\beta}$, $\alpha, \beta, \gamma = 1, \ldots, n^2 - 1$, are structure constants of su(n) with respect to the basis (e_α), then $c^\gamma_{\alpha\beta}$ are also structure constants of the Lie algebra $\mathrm{Der}(M_n(\mathbf{C})$ with respect to the basis $(\mathrm{ad}e_\alpha)$, $\alpha = 1, \ldots, n^2 - 1$. In our case, we choose the following basis for su(2):

$$e_1 = \mathrm{ad}\,\frac{i}{2}\,\sigma_1, \qquad e_2 = \mathrm{ad}\,\frac{i}{2}\,\sigma_2, \qquad e_3 = \mathrm{ad}\,\frac{i}{2}\,\sigma_3$$

where $\sigma_1, \sigma_2, \sigma_3$ are the usual Pauli matrices, and the structure constants assume the simple form $c^\gamma_{\alpha\beta} = \epsilon^{\alpha\beta\gamma}$, i.e., they are equal to 1 for even permutations, and 0 otherwise (e.g., Choquet-Bruchat *et al.*, 1982).

Now we choose the metric $g_{D_4}: V_{D_4} \times V_{D_4} \to \mathcal{Z}(\mathbf{C}[D_4])$. Since the metric $pr^*_{D_4} g_{D_4}$ is essentially unique, we choose the metric g_{D_4} in the simplest possible form,

$$(g_{D_4})_{ij} = \delta_{ij}\mathbf{I}$$

and develop differential geometry as in Section 3.

Straightforward computations give the following nonvanishing components of the Ricci operator:

$$R^1_1 = R^2_2 = R^3_3 = \frac{1}{2}$$

It can be easily seen that in this case the Einstein equation $\mathbf{R}_{D_4}(w) = 0$ is satisfied only for $w = 0$, $w \in V_{D_4}$. To have nontrivial solutions we should try the Einstein equation with the cosmological constant. For $w = w^i v_i$, it can be written in the form

$$(\mathbf{R}_{D_4} + 2\Lambda \mathrm{id}) w^i v_i = 0$$

This is the eigenvalue equation for the operator $\mathbf{R}_{D_4}$ with the eigenvalues $\lambda = 2\Lambda$. One can easily find that $\Lambda = 1/4$, and correspondingly one has

$$\ker(\mathbf{G}_{D_4}) = \{tv_1 \colon t \in \mathbf{C}\}$$

It is a remarkable fact that our simple model requires for its consistency the existence of the cosmological constant in the quantum sector of the model (i.e., in its "vertical geometry"), and predicts its value. Of course, because of the "toy" character of the model, this value is rather symbolic.

Now we must return to the "parallel geometry." We shall consider the algebra $\mathcal{A}_E = \pi_2^*(C^\infty(E))$, where π_2 is the natural projection from $G = E \times D_4$ to E. Since in our case the frame bundle $\pi_1 \colon E \to M^3$ is a trivial bundle, there is a natural embedding $\iota_\kappa \colon M \to E$, $\kappa \in \Gamma$, where Γ is the structural group of this bundle, i.e., the Lorentz group. There is also another natural embedding $j_h \colon E \to G$, $h \in D_4$. With the help of these two embeddings we "push forward" the basis $(\partial_t, \partial_x, \partial_y)$ in M^3 to the basis $(\bar{\partial}_t, \bar{\partial}_x, \bar{\partial}_y)$ in G. In this way, we obtain the $\mathcal{Z}(\mathcal{A})$-module of derivations

$$V_E = \{\alpha^1\bar{\partial}_t + \alpha^2\bar{\partial}_x + \alpha^3\bar{\partial}_z \colon \alpha^1, \alpha^2, \alpha^3 \in \mathcal{A}_E\}$$

We equip this module with the metric $\bar{\eta}$ lifted from the Minkowski metric η on M^3, i.e., $\bar{\eta} = \tau^*\eta$, where $\tau = \pi_1 \circ \pi_2$. It can be easily seen that in fact $\bar{\eta}$ is also the Minkowski metric (it can be weakly degenerate; see Section 2.2). Indeed, let $\bar{X}, \bar{Y} \in V_E$; then $\bar{\eta}(\bar{X}, \bar{Y}) = (\tau^*\eta)(\bar{X}, \bar{Y}) = \eta(\tau_*\bar{X}, \tau_*\bar{Y}) = \eta(X, Y)$.

Therefore, our model is described by the algebra $\mathcal{A} = C^\infty(E \times D_4)$ together with the $\mathcal{Z}(\mathcal{A})$-module of derivations of $\mathcal{A}$

$$V = V_E \oplus V_{D_4} = \{\alpha^a\bar{\partial}_a + \beta^i\bar{\upsilon}_i \colon \alpha^a, \beta^i \in \mathcal{A}, a = 0, 1, 2; i = 1, 2, 3\}$$

Now we should see how the generalized Einstein equation interacts with the quantum dynamical equation (5). Since we postulate that the derivation v in the left-hand side of equation (5) should be a solution of the generalized Einstein equation, equation (5) assumes the form

$$i\hbar\pi_q\left(\left(\alpha^a\bar{\partial}_a + t\,\mathrm{ad}\,\frac{i}{2}\,\sigma_1\right)(a)\right) = [\pi_q(a), F]$$

and, for $a \in \mathcal{A}_{\mathbf{G}}$, $[\pi_q(a), F] = 0$ (see Section 5).

In agreement with the discussion of Section 5, in order to obtain the classical case we must restrict the algebra $\mathcal{A} = C^\infty(G)$ to the algebra $\mathcal{A}_{\text{proj}}$, and the proof that $[\pi_q(a), F] = 0$ for all $a \in \mathcal{A}_{\mathbf{G}}$ is no longer valid. In such a case, the "vertical geometry" projects to zero, and we are left with the ordinary Minkowski spacetime M^3 and the corresponding Einstein field equation $\mathbf{R} = 0$ (this effect, in the considered model, is trivial since the "parallel geometry" has been obtained by lifting the Minkowski geometry to the groupoid G). Let us notice that, even in this toy model, we have an interesting result: in the noncommutative regime a kind of cosmological constant appears (as an eigenvalue of the Ricci operator for the "quantum sector") which vanishes if we go to the classical case.

7. OBSERVABLES AND THEIR EIGENVALUES

In postulate 1 of our quantization scheme we identified quantum gravity observables with the Hermitian elements of the algebra $\mathcal{A}$ by following strict analogy with the C^*-algebraic quantization in the usual quantum mechanics. However, from the experimental point of view we are interested only in those observables which leave some traces in the macroscopic world and thus have a chance to be detected. As seen in the preceding section, such observables must belong to $\mathcal{A}_{\text{proj}}$. Let a be such an observable, and let the system be in a state ψ which, in order "to be reached" by a macroscopic observer, must be Γ-invariant (see Section 5). Measuring an observable quantity corresponding to a when the system is in a Γ-invariant state $\psi \in L^2(G_q)$ means acting with a upon ψ. The measurement will give as its result the eigenvalue r_q as determined by the eigenvalue equation

$$\pi_q(a)\psi = r_q\psi \tag{7}$$

where, for simplicity, we consider a nondegenerate case. Taking into account the form of the representation π_q [equation (4)], the above equation is equivalent to

$$\int_{G_q} a(\gamma_1)\psi(\gamma_1^{-1}\gamma) = r_q\psi(\gamma)$$

From the Γ invariance of ψ it follows that ψ is constant on G_q; therefore, we can write

$$\psi(\gamma_1^{-1}\gamma)\int_{G_q} a(\gamma_1) = r_q\psi(\gamma)$$

and consequently

$$r_q = \int_{G_q} a(\gamma_1)$$

Therefore we have proved the following fact: *If $\psi \in L^2(G_q)$ is Γ-invariant and if it is an eigenfunction of $a \in \mathcal{E}_H$, the eigenvalue of a is $r_q = \int_{G_q} a(\gamma_1)$*. This is a nice conclusion. Let us notice that the result r_q of a measurement is a measure in the mathematical sense (in this case we deal with the Haar measure on the group Γ). But we can go even further. Let us define the "total phase space" of our system

$$L^2(G) := \bigoplus_{q \in G^{(0)}} L^2(G_q)$$

with the operator $\pi(a) := (\pi_q(a))_{q \in G^{(0)}}$ acting on it. Now, the eigenvalue equation (7) can be naturally written as $\pi(a)\psi = r\psi$, where r is a function on G; since, however, ψ is Γ-invariant, r can be interpreted as the function on spacetime M

$$r: \quad M \to \mathbf{R}$$

defined by $r(x) = r_q = \int_{G_q} a(\gamma_1)$ with x being a point in M to which the "frame" q is attached. The measurement result is not a "naked number," but a value of a function $r(x)$ at a given point $x \in M$ the domain of which is the entire spacetime M. As the consequence of this, if the measurement is performed at a point $x \in M$, its result $r(x)$ is correlated with the result $r(y)$ of another measurement of the same kind, performed at another point $y \in M$, on another component of the same system even if points x and y are far from each other. This also suggests that typically quantum gravitational phenomena should be looked for among correlations between distant measurements rather than among "local phenomena." This could be regarded as a relic of the pre-Planckian era in which "everything was global."

So far we have been interested in what happens when we project the algebra $\mathcal{A}$ onto the "horizontal component" E of the groupoid G. This gives us the transition to the classical spacetime geometry (general relativity). Now we shall study the projection of $\mathcal{A}$ onto the "vertical component" Γ of G. This will give us quantum effects of our model. In analogy with $\mathcal{A}_{\text{proj}}$, let us define the subalgebra $\mathcal{A}_\Gamma$ of functions *projectible* to Γ

$$\mathcal{A}_\Gamma := \{f \circ pr_\Gamma : f \in C_c^\infty(\Gamma, \mathbf{C})\} \subset \mathcal{A}$$

Let $a \in \mathcal{A}_\Gamma$, and let us consider the following representations of $\mathcal{A}_\Gamma$ [see equation (4)]:

$$\pi_p(a)(\xi_p) = a_p * \xi_p \tag{8}$$

and

$$\pi_q(a)(\xi_q) = a_q * \xi_q \tag{9}$$

where $\xi_p \in L^2(G_p)$, $\xi_q \in L^2(G_q)$, p, $q \in E$, $p \neq q$. Since G_p and G_q are isomorphic, we can choose ξ_p and ξ_q to be isomorphic with each other. This in turn implies that $\pi_p(a)$ and $\pi_q(a)$ are isomorphic as well. In this way we have obtained the following important result: *If $a \in \mathcal{A}_\Gamma$, then its image under the representation π_p does not depend on the choice of $p \in E$ (up to isomorphism)*. In simple terms: all points of M "know" what happens in any fiber G_g, $g \in \Gamma$. This opens the way to explaining nonlocal phenomena, such as the Einstein–Podolsky–Rosen experiment, in quantum mechanics.

8. EINSTEIN–PODOLSKY–ROSEN EXPERIMENT FROM NONCOMMUTATIVE GEOMETRY

In this section we shall show that nonlocal effects met in quantum mechanics can be regarded as "shadows" from the "totally nonlocal" physics of the Planck regime [for details see Heller and Sasin (1998)]. Let us assume that $\Gamma_0 = SU(2)$ is a subgroup of Γ. We look for an element $s \in \mathcal{A}_\Gamma$ such that $\pi_p(s)$: $L^2(\Gamma_0) \to L^2(\Gamma_0)$. We define two linearly independent functions on Γ_0, the constant functions $\mathbf{1}$: $\Gamma_0 \to \mathbf{C}$ and det: $\Gamma_0 \to \mathbf{C}$, which span the linear space $\mathbf{C}^2 \subset L^2(\Gamma_0)$, i.e., $\mathbf{C}^2 = \langle \mathbf{1}, \det \rangle_{\mathbf{C}}$. Let $\hat{S}_z = \pi_p(s)|_{\mathbf{C}^2}$ be the z component of the usual spin operator. We have

$$\pi_p(s)\psi = \hat{S}_z\psi$$

for $\psi \in \mathbf{C}^2$ or, by using representation (4) and taking into account that $\hat{S}_z\psi = \pm(\hbar/2)\psi$, we obtain

$$\int_{\Gamma_0} s_p(\gamma_1)\psi(\gamma_1^{-1}\gamma) = \pm\frac{\hbar}{2}\psi$$

and since $s_p = \text{const}$,

$$\int_{\Gamma_0} \psi(\gamma_1^{-1}\gamma) \sim \psi(\gamma)$$

One of the solutions of this equation is $\psi = \mathbf{1}_{\Gamma_0}$. Hence,

$$\frac{\hbar}{2} = \pm\int_{\Gamma_0} s_p(\gamma_1)$$

which gives

$$(s_p)_1 = +\frac{\hbar}{2}\frac{1}{\text{vol}\Gamma_0}$$

$$(s_p)_2 = -\frac{\hbar}{2}\frac{1}{\text{vol}\Gamma_0}$$

Therefore, the corresponding eigenvalue equations are

$$\pi_p((s_p)_1)\psi = +\frac{\hbar}{2}\psi \qquad \text{for} \quad \psi \in \mathbf{C}^+$$

$$\pi_p((s_p)_2)\psi = -\frac{\hbar}{2}\psi \qquad \text{for} \quad \psi \in \mathbf{C}^-$$

where $\mathbf{C}^+ := \mathbf{C} \times \{0\}$ and $\mathbf{C}^- := \{0\} \times \mathbf{C}$.

Now, the "EPR paradox" can be discussed in the standard way. Let A be an observer situated at $\pi_M(p) = x_A \in M$ who measures the z-spin component of the one of the electrons, i.e., he applies the operator $\hat{S_z} \otimes \mathbf{1}|_{\mathbf{C}^2}$ to the vector $\xi = (1/\sqrt{2})(\psi \otimes \varphi - \varphi \otimes \psi)$, where $\psi \in \mathbf{C}^+$ and $\varphi \in \mathbf{C}^-$. Let us assume that as a result he obtains the value $(\hbar/2)$. Therefore, as the effect of the measurement, the state vector

$$\xi = \frac{1}{\sqrt{2}}(\psi \otimes \varphi - \varphi \otimes \psi) \in \mathbf{C}^2 \otimes \mathbf{C}^2 \subset L^2(G_r) \otimes L^2(G_r), \qquad r \in E$$

collapses to $\xi_0 = (1/\sqrt{2})(\psi \otimes \varphi)$. Therefore, immediately after the measurement the system is in the state ξ_0, but this is the same (up to isomorphism) for all fibers G_r, $r \in E$. Let us notice that formulas (8) and (9) are obviously valid also for tensor products, and consequently the state ξ_0 does not depend on the point in spacetime to which the frame r is attached. In particular, the vector ξ_0 is the same for the fibers G_p and G_q, where p is such that $\pi_M(p) = x_A$ and q is such that $\pi_M(q) = x_B$ ($x_A \neq x_B$). Now, if observer B situated at x_B measures the z-spin component of the second electron, i.e., if he applies the operator $\mathbf{1}|_{\mathbf{C}^2} \otimes \hat{S_z}$ to the vector ξ_0, he must obtain the value $-\hbar/2$ as the result of his measurement.

Since the main tool in deducing nonlocal phenomena, and in particular the EPR effect, from our model is projecting from G to either E or Γ, we can truly say that nonlocal phenomena, actually known in quantum mechanics, are shadows of the noncommutative regime reigning at the fundamental level.

9. CONCLUDING REMARKS

In the present paper we have proposed a mathematical structure which combines essential aspects of general relativity (its main geometric elements) with those of quantum physics (algebra of operators on a Hilbert space), and leads to correct "special cases" of standard general relativity and standard quantum mechanics. This result is rooted in the fact that we have performed the quantization of a groupoid (over a spacetime) rather than of spacetime itself.

Although our approach is a scheme rather than a full theory [since two of its important elements, the group Γ and the operator F in equation (5), remain unspecified], it gives some hints of what the future theory, based on a noncommutative theory, could be like. It also clearly suggests that measurable effects of quantum gravity should be looked for among correlations of distant phenomena rather than among local effects (Section 7). The calculation of the Einstein–Podolsky–Rosen effect (Section 8) not only strongly confirms this suggestion, but also could be regarded as a "retrodiction" confirming (at least some aspects of) our model. It is also a remarkable fact that a simple model based on the groupoid $G = E \times D_4$ (Section 6) predicts the value of the "cosmological constant" which, after projecting down to spacetime, vanishes.

To choose the correct group Γ it is important to experiment with various possibilities. In Heller *et al.* (1997) we computed an example where Γ is any finite group. The next should be the generalization to the case when Γ is compact. However, from the physical point of view the most interesting are cases with Γ noncompact; in these cases, instead of the algebra $\mathscr{A} = C_c^\infty(G,\mathbf{C})$ one could consider the algebra $\mathscr{A}_0 = C_o^\infty(G,\mathbf{C})$ of smooth functions vanishing at infinity [$\mathscr{A}$ is the norm closure of $\mathscr{A}_0$; see Landi (1997), p. 168]. In such cases, one would have to deal with families of noncommutative algebras rather than with a single algebra. For instance, as the work by Fell (1961) demonstrates, the group algebra of SL(2,C) (which physically is very interesting) can be expressed in terms of algebras of operator fields on a locally compact Hausdorff space, which in turn is isomorphic with the algebra of norm-continuous functions on a certain parameter space.

REFERENCES

Chamseddine, A. H., and Connes, A. (1996a). The spectral action principle, Preprint hep-th/9606001.

Chamseddine, A. H., and Connes, A. (1996b). A universal action formula, Preprint hep-th/9606056 (1996).

Chamseddine, A. H., Felder, G., and Fröhlich, J. (1993). *Commun. Math. Phys.* **155**, 205–217.

Choquet-Bruchat, Y., DeWitt-Morette, C., and Dillard-Bleick, M. (1982). *Analysis, Manifolds and Physics*, rev. ed. North-Holland, Amsterdam.

Connes, A. (1979). Sur la theorie non commutative de l'intgration, in: *Algèbres d'Operateurs*, Lecture Notes in Mathematics, No. 725, Springer, Berlin, pp. 19–143.

Connes, A. (1994). *Noncommutative Geometry*, Academic Press, New York.

Connes, A. (1996). *Commun. Math. Phys.* **182**, 155–176.

Connes, A., and Lott, J. (1990). *Nucl. Phys. Suppl.*, *B*, **18**, 29–47.

Connes, A., and Rovelli, C. R., (1994). *Class. Quant. Grav.* **11**, 2899–2917.

Demaret, J., Heller, M., and Lambert, D. (1997). *Found. Sci.* **2**, 137–176.

Dubois-Violette, M. (1988). *C. R. Acad. Sci. Paris* **307**, 403–408.

Dubois-Violette, M. (1995). Some aspects of noncommutative differential geometry, preprint q-alg/9511027.

Dubois-Violette, M., and Michor, P. W. (1994). *C. R. Acad. Sci. Paris* **319**, (*Ser. I*), 927–931.

Dubois-Violette, M., Kerner, R., and Madore, J. (1989a). *Phys. Lett. B* **217**, 485.

Dubois-Violette, M., Kerner, R., and Madore, J. (1989b). *Class. Quant. Grav.* **6**, 1709–1724.

Dubois-Violette, M., Kerner, R., and Madore, J. (1990). *J. Math. Phys.* **31**, 316–322.

Fell, J. M. G. (1961). *Acta Math.* **106**, 233–280.

Hajac, P. M. (1996). *J. Math. Phys.* **37**, 4549–4556.

Heller, M., and Sasin, W. (1995). *J. Math. Phys.* **36**, 3644–3662.

Heller, M., and Sasin, W. (1996). *J. Math. Phys.* **37**, 5665–5671.

Heller, M., and Sasin, W. (1997). *Banach Center Publications* **41**, 153–162.

Heller, M., and Sasin, W. (1998). Einstein–Podolsky–Rosen experiment from noncommutative quantum gravity, in: *Particles, Fields, and Gravitation,* ed. by J. Rembieliński, American Institute of Physics, Woodbury-New York, 234–241.
Heller, M., Sasin, W., and Lambert, D. (1997). *J. Math. Phys.* **38**, 5840–5853.
Koszul, J. L. (1960). *Fibre Bundles and Differential Geometry*, Tata Institute of Fundamental Research, Bombay.
Landi, G. (1997). *An Introduction to Noncommutative Spaces and Their Geometry*, Springer, Berlin.
J. Madore, (1995). *An Introduction to Noncommutative Differential Geometry and Its Physical Applications* , Cambridge University Press, Cambridge.
Madore, J. (1997). Gravity on fuzzy space-time, ESI Preprint 478.
Madore, J., and Mourad, J. (1996). *J. Math. Phys.* **39**, 423–442.
Masson, T. (1996). Géometrie non commutative et applications à la théorie des champs, Thesis, preprint ESI 296, Vienna.
Palais, R. S. (1981). *Real Algebraic Differential Topology*, Publish or Perish, Wilmington, Delaware.
Renault, J. (1980). *A Groupoid Approach to C* Algebras*, Lecture Notes in Mathematics, No 793, Springer, Berlin.
Rovelli, C. (1990). *Phys. Rev. D* **42**, 2638–2646.
Rovelli, C. (1991). *Phys. Rev. D* **43**, 442–456.
Sasin, W., and Heller, M. (1995). *Acta Cosmol.* (Cracow) **21**(2), 235–245.
Serre, P. (1978). *Représentations linéaires des groupes finis*, Hermann, Paris.
Sitarz, A. (1994). *Classical and Quantum Gravity* **11**, 2127–2134.
Thirring, W. (1979). *Lehrbuch der Mathematischen Physik*, Vol. 3, Springer, Vienna.